THE MEASUREMENT THEOREMS IN GEOMETRY

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In the presentation of the idea of measurement in geometry there are two difficulties. One arises from the necessity of treating incommensurable quantities; the other is due to the totally unpractical point of view from which the subject is approached. The beginner in the study of demonstrative geometry has no knowledge of incommensurable quantities until they are spoken of in connection with this subject; he has trouble in realizing the nature of the obstacle he is being taught to surmount. As to the other difficulty, that the treatment of measurement in geometry is unpractical, remember that the pupil has already computed areas of rectangles, and that in general he has had practice in the numerical estimation of a great many different kinds of quantity. In all such practice he has learned to compare quantities by means of the numbers that represent them, and he perfectly well understands that these numbers are obtained by comparison with standard units, or with fractions of those units that are obtained by a standard scheme of subdivision.

In his study of geometry, however, this knowledge is of no use to him. He is given a pair of rectangles, first with equal bases and then without, but having no relation whatever to a standard unit; he is taught to compare them by a cumbrous method, especially invented for the purpose, and having apparently nothing to do with measurement as he has known it and as everybody knows it. After he has worried through this, he learns that if one of his rectangles degenerates into a unit square, his investigation has proved that the area of the other rectangle is the product of the lengths of its sides; and this, the main object of his work, instead of being the center of his interest, comes in as an anticlimax.

The unsatisfactory results of these methods are well known; perhaps they are unavoidable. It seems worth while, however, to try a

method in which the pupil will not have to accept the existence of incommensurable numbers before entering upon the line of investigation that explains their origin; in which he investigates measurement in geometry just as he would practice it with the instruments of the carpenter, the engineer, or the astronomer; and in which thoroughgoing proofs are given of the central facts of measurement, by direct attack and without any sacrifice of validity.

The theorems in elementary geometry that form the logical basis for the recognition of a correspondence between number and geometrical magnitude are these:

A central angle is measured by its arc.
A diedral is measured by its plane angle.
The area of a rectangle is equal to the product of its base and altitude.
The volume of a parallelopiped is equal to the product of its three dimensions.

On the other hand, the practical basis for this recognition is experience in measurement. The measurement of an angle with a protractor, and of floor areas with a tape or a yard-stick, leads at once to associations of number and magnitude—associations that are familiar to every pupil, a part of the mental stock in trade which the teacher of geometry should utilize, if possible. It can be utilized, without any sacrifice in mathematical rigor, and with some advantage in directness, in the demonstration of the four theorems referred to above.

In the preliminary study of the measurement of a straight line, emphasis should be laid on the measurements of the engineer, rather than on those of the carpet-man. The exactness obtainable by great care, the use of verniers, micrometers, scale-reading microscopes, the allowance for temperature, the devices for keeping a measuring-tape under constant tension—all these things will add interest to such a simple topic as the approximate measurement of a straight line. They will also tend to fix in the pupil's mind the decimal scales of measurement, which will be a further advantage in the effort to associate number with magnitude.

This limitation to decimal numbers furnishes an introduction to the study of incommensurable numbers. In the entire series of decimal numbers, infinite though it is, from 0 to 1, there is none that is exactly equal to $\frac{1}{3}$; and in the entire infinite series of distances obtained from the unit by decimal subdivision, there is none that, repeated

three times, will give the original unit. It is, however, easy to find two numbers, each with its corresponding distance, one larger and the other smaller than the number sought, and differing from each other by an amount smaller than any assigned number, however small that may be; and the corresponding distances will also be one larger and the other smaller than the required distance; moreover, they will differ from each other by an amount smaller than any assigned distance, however small that may be. Consequently, although with our limitation to decimals we cannot actually identify this required distance and its corresponding number, we can come as close to it, from above or below, as we choose; that is, we can find a distance, and a corresponding number, which differ from those required by less than $\frac{1}{n}$ of the unit, where n is any assigned power of 10 whatever.

The two theorems upon which is based the measurement of angles and arcs are these:

In equal circles, if two central angles are equal, their arcs are.

In equal circles, if two arcs are equal, their central angles are.

As we measured distances by successive applications of a standard distance, so we measure arcs by successive applications of a standard arc, and angles in the same way, using for a standard angle the central angle that intercepts the standard arc. From the theorems above it is evident that for arcs that can be exactly measured by the degree, which serves as a unit, or by its subdivisions (minutes, seconds, and decimals of a second), the number that represents the size of the angle is identical with the number that represents the arc.

As in the case of a rectilinear distance, it is further evident that there are arcs (for example, one-seventh of a quadrant) which are not represented by any number in the infinite series of numbers that can be obtained by decimal subdivision of the second, however far that subdivision may be carried. And as in that case, one may obtain an arc which differs from the required arc by less than $\frac{1}{n}$ of a second, where n is any stated power of 10, however large. This arc, and its angle, are represented by the same number; then the required arc, and its angle, are represented by numbers which cannot differ by any stated amount, however small.

The same argument can be applied to any incommensurable arc, and its angle; for example, to the arc equal in length to the radius. The advantage of applying this argument first to arcs that are commensurable, though not decimally expressible, is that with this plan it is not necessary for the pupil to accept at the beginning such a radically new idea as incommensurable number.

The conclusion, that an arc and its central angle are expressed by the same number, is briefly stated thus: A central angle is measured by its intercepted arc.

Precisely analogous treatment is given to the theorem about diedrals (and lunes).

The area of a rectangle, in the case where the lengths of the sides can be exactly measured by the unit of length or by any of its subdivisions, is evidently the number that represents the product of the number of squares in a row by the number of rows, divided, if necessary, by a number that represents the extent to which subdivision is carried. In order that the fractional unit-areas may be squares, it is necessary for each side to be expressed to the same number of decimal places (using ciphers if necessary). If the sides of a rectangle, for example, are 2.150 and 4.325, the area will be represented by the number of squares of area 0.000001 of the square unit.

Rectangles may be found of which the sides cannot be expressed by a number of the decimal system. Measuring two sides from the same corner, and drawing parallels through the last mark of exact measurement, a new rectangle is obtained the area of which is exactly expressible by a decimal number which is the product of the numbers that express the lengths of the sides.

This new rectangle we will call the *approximate rectangle*, and the length numbers of its sides we will represent by x and y; the length numbers of the corresponding sides of the original rectangle, whatever these numbers are, we will represent by a and b. As we have seen, the values of x and y may be made to approach as nearly as we please to a and b.

However far we carry our decimal subdivision of the unit of length, the approximate rectangle will differ from the original rectangle by a strip which extends around two sides of the rectangle, and which has

a width less than $\frac{1}{n}$ of the unit of length. This strip we may call the *error of area.*

In the accompanying diagrams, Fig. 1 represents the approximate rectangle divided into squares each side of which is $\frac{1}{n}$ of the unit of length. The error of area is represented by a shaded band. In Fig. 2 the error of area is straightened out, so as to show that it is less than a rectangle whose length is $x+y+\frac{1}{n}$, and whose width is $\frac{1}{n}$. The pupil will need to be cautioned against the mistake of considering the error of area as necessarily of uniform width. This caution may be emphasized by magnifying the diagram: for example, Fig. 3 represents Fig. 1

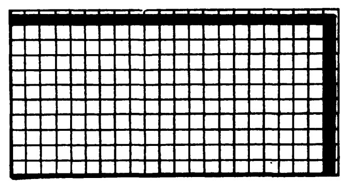

FIG. 1

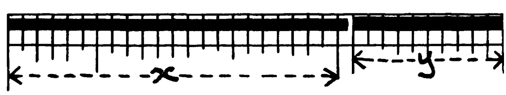

FIG. 2

magnified $\times 4$, and the difference in width between the two strips is obvious. Fig. 4 is a similar enlargement of Fig. 2.

There is a strip of uniform width contained between the approximate rectangle and another rectangle, formed by drawing parallels through the next measurement marks, outside the original rectangle. This strip has an area $\frac{1}{n}\left(x+y+\frac{1}{n}\right)$, where n is the number of subdivisions of the unit of length; and would exceed the error of area at every stage of the approximation.

Now, the subdivision of the unit of length can theoretically be carried as far as we choose; that is, we can make $\frac{1}{n}$ smaller than any stated number whatever. Then the error of area, which is less than an area expressed by $\frac{1}{n}\left(x+y+\frac{1}{n}\right)$, may be made less than any stated area, however small; and thus we have proved that the theorem cannot be in error by any stated amount, however small.

In practical approximation this theory can be turned to account as follows: Suppose a rectangle is roughly 27 ft. by 68 ft. and it is required to determine its area to tenths of a square foot. The expression for the error of area then will give the equation

$$\frac{1}{n}\left(95+\frac{1}{n}\right)=\frac{1}{10}\ ;$$

whence

$$950+\frac{10}{n}=n\ ;$$

and from this it is clear that for the required degree of accuracy the sides should be measured to thousandths of a foot.

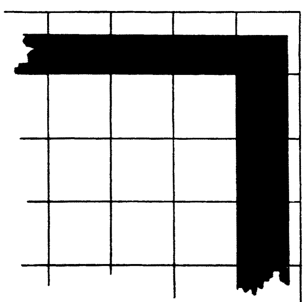

FIG. 3

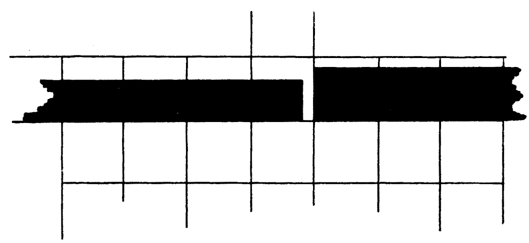

FIG. 4

Again, suppose a surveyor is able to measure to $\frac{1}{100}$ of a foot; what error is possible in a rectangular lot 300 ft. by 200 ft.? The formula gives

$$\frac{1}{100}(300+200+0.01)$$

which reduces to an error of 5 sq. ft. out of 60,000—a result which leaves out of account the fraction $0.0001=\frac{1}{n^2}$. Here is the first appearance to the pupil of a quantity of the second order of littleness; whether he goes into college mathematics or not, this idea is of great importance to him.

In the same way for the measurement of a rectangular block we invent an *approximate* solid, and the remainder of the solid to be measured is always less than a layer of uniform thickness, spread over three faces of the approximate solid, and in thickness less

than the finest subdivision (of the unit of length) that has been used for measuring the edges. The *error of volume*, then, is less than

$$\frac{1}{n}(xy+yz+zx)+\frac{1}{n^2}(x+y+z)+\frac{1}{n^3} \ ,$$

where x, y, and z are the length-numbers of the three different edges of the approximate solid, and n is the number of subdivisions of the unit of length.

For example, suppose a rectangular bin is measured in meters 1.50, 2.00, and 3.20. Its volume then would be 9.600 cu. m. If the measurement were correct to hundredths of a meter, the error could not be so great as

$$0.142+0.0067+0.000001 \text{ cu.m.} \ ;$$

and if the measurements had been carried to millimeters the error would be less than

$$0.0142+0.000067+0.000000001 \text{ cu.m.} \ ,$$

the relative importance of the three terms of the expression being clearly brought out as the degree of accuracy progresses.

It may be objected that this demonstration is complicated. To be sure it is. The truth which it establishes is one of the most easily believed in the whole of geometry, and one of the hardest to prove to young pupils. The usual demonstration for it is complicated, as are all other propositions about incommensurables, by dependence upon the following "theorem of limits:"

If two variable quantities are necessarily represented by the same number, and if each of the variable quantities approaches a limit, the limits are represented by the same number.

There is, however, an appearance of simplicity in the usual demonstration, on account of the separate preliminary consideration of this theorem. The objections to the treatment of the theorem of limits as a preliminary, a lemma, to the measurement theorems, are: first, that the pupil will not know what his lemma means until he gets to the measurement theorems; second, that the theorem of limits does not need any proof: as soon as the pupil can understand what it means he will see that no proof is needed.

The theorem of limits is applied also to the demonstration of another important proposition, for which the device suggested in this article, namely, measurement by a standard unit, will not serve:

A parallel to the base of a triangle divides the two sides proportionally.

Upon this theorem rests the whole theory of similar triangles, and if we had to abandon here our principle of always measuring by a standard unit, reverting for this case only to the arbitrary special devices discarded in the measurement propositions, the surrender would make our plan of reform hardly worth considering. But no such abandonment is necessary. The theorem quoted can be made to depend on the measurement theorems, the line of descent being as follows:

Triangles having constant bases vary as their altitudes.

Triangles having an angle constant vary as the product of the sides including the constant angle.

Equiangular triangles have their sides proportional.

It is needless to say that this suggestion will involve a change in the order of propositions that has become traditional in this country; but the logical order of propositions that happens to be in vogue at our examination stations has no sanctity that a teacher is bound to respect. "The subject, not the book," is expected.